Statistical Moments

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In statistics and probablity, moments are a set of mathematical measurements used to describe center, shape and spread of distribution $X = \{x_1, x_2, x_3, ..., x_n\}$ is the sample set with n values. In the following text, $\langle .. \rangle$ is the average of the data points in the angle brackets.

1. 1st Moment: Mean (μ). 'Center of gravity' of the distribution. for n data points in sample set X:

$$\mu = \frac{1}{n} \sum_{i=1}^{n} x_i = \langle X \rangle \tag{1}$$

If the mean shifts the entire distribution shifts. It is common practice to subtract the mean from a distribution to center the curve at zero.

2. 2nd Moment: Variance (σ^2). Measure of the spread of data points around the mean. Standard deviation is σ.

$$\sigma^2 = \langle (X - \mu)^2 \rangle \tag{2}$$

High variance indicates that the distribution is spread out from the center/mean; low variance means the data points are clusters near the mean value. Interestingly, as the variance approaches zero, the probability density function $f(x)$ becomes a Dirac delta function: $\lim_{\sigma \to \mathbf{0}} \mathbf{f}(\mathbf{x}) = \delta(\mathbf{X} - \mu)$.

3. 3rd Moment: Skewness (Asymmetry; γ). Measures the lack of symmetry in distribution.

$$\gamma = \left\langle \left(\frac{X - \mu}{\sigma} \right)^3 \right\rangle \tag{3}$$

Zero skewness means perfectly symmetric (normal distribution). Positive or negative skewness mean right side or left side respectively is longer (or fatter).

4. 4th Moment: Kurtosis (β) is the fourth standardized moment. Describes the 'tailed-ness'; of distribution; meaning how much of the data resides in the tail vs the peak.

$$\beta = \left\langle \left(\frac{X - \mu}{\sigma} \right)^4 \right\rangle \tag{4}$$

For kurtosis, $k = 3$ is a special value:

$$k \begin{cases} = 3 & \text{Normal distribution.} \\ > 3 & \text{Sharp peak with heavy tails. More outliers. Thin shoulders (around } \sim 1\sigma). \\ < 3 & \text{Flat peak with thin tails. Few outliers. Beefy shoulders.} \end{cases}$$

Normal Distribution

Normal distribution or Gaussian distribution or the probability distribution function (PDF) is given by:

$$f(x) = N(\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2\right].$$

Standard normal distribution (Z) is when mean is zero and standard deviation is 1: $Z \sim N(\mu = 0,\ \sigma = 1)$. It is given by:

$$Z(\mu = 0, \sigma = 1) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}X^2\right).$$

PDF gives us the points along the distributions. Cumulative distribution function (CDF) gives us the probability of finding a point greater than or less than a value 'x' and is given by:

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$$

As a result we get: For a normal distribution, the probability of a data point lying within one standard deviation ($\sigma = 1$) of the mean is approximately 68.27%. Similarly, the probabilities within 2σ and 3σ are 95.45% and 99.73%, respectively.