Stability proof for the explicit finite-difference scheme of the 1D heat and wave equation

Rahul Pandare | November 12, 2025

1 For 1D heat equation[4, 1]

1.1 PDE and discretization

Consider the one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \qquad \alpha > 0,$$

on a spatial domain (for example $x \in [0, L]$) with boundary conditions and an initial condition $u(x, 0) = u_0(x)$.

Introduce a uniform spatial grid $x_i = i\Delta x$ and time levels $t^n = n\Delta t$. Let u_i^n denote the numerical approximation to $u(x_i, t^n)$. Using the forward difference in time and the central difference in space (the FTCS scheme[3]), the discrete update reads

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}.$$

Rearranging,

$$u_i^{n+1} = u_i^n + r\Big(u_{i+1}^n - 2u_i^n + u_{i-1}^n\Big), \qquad r := \frac{\alpha\,\Delta t}{(\Delta x)^2}. \tag{1}$$

The dimensionless parameter r is often called the Fourier (or diffusion) number.

1.2 von Neumann (Fourier) stability analysis

To assess linear stability of the scheme (1), we perform a von Neumann analysis. Assume the numerical error (or a Fourier mode of the solution) can be written as a single Fourier mode

$$u_i^n = G^n e^{\mathrm{i}kx_i},$$

where k is the wave number, G is the amplification factor (generally complex), and $\mathrm{i} = \sqrt{-1}$. Substitute this ansatz into the update (1). Using $x_i = i\Delta x$,

$$G^{n+1}e^{\mathrm{i}kx_i} = G^n e^{\mathrm{i}kx_i} + r\Big(G^n e^{\mathrm{i}kx_{i+1}} - 2G^n e^{\mathrm{i}kx_i} + G^n e^{\mathrm{i}kx_{i-1}}\Big).$$

Divide both sides by $G^n e^{\mathrm{i}kx_i}$ (assuming $G \neq 0$):

$$G \;=\; 1 + r\big(e^{\mathrm{i}k\Delta x} - 2 + e^{-\mathrm{i}k\Delta x}\big).$$

Using $e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta} = 2\cos\theta$, we get

$$G = 1 + r\big(2\cos(k\Delta x) - 2\big) = 1 - 2r\big(1 - \cos(k\Delta x)\big).$$

Because $1 - \cos\theta = 2\sin^2(\theta/2)$, this becomes

$$G = 1 - 4r\sin^2\big(\tfrac{k\Delta x}{2}\big). \tag{2}$$

Observe that for this scheme G is *real* (no imaginary part) and depends on $k\Delta x$ only through $\sin^2$. For stability, every Fourier mode must not grow in magnitude with time; i.e. we require

$$|G| \leq 1 \qquad \text{for all wave numbers } k.$$

1.3 Determining the stability condition

From (2) we have G real. The largest possible magnitude of the decrement $4r\sin^2(\frac{k\Delta x}{2})$ occurs when $\sin^2(\frac{k\Delta x}{2})$ is maximal, i.e. equals 1. Hence the most restrictive case is

$$G_{\min} = 1 - 4r.$$

Requiring $|G| \le 1$ for all modes implies both

$$G \le 1 \quad (\text{always true here since } \sin^2 \ge 0)$$

and

$$G \ge -1.$$

Thus the stability requirement reduces to

$$1 - 4r \ge -1 \quad\Longrightarrow\quad -4r \ge -2 \quad\Longrightarrow\quad r \le \tfrac{1}{2}.$$

Additionally, because $\Delta t \ge 0$ and $\alpha > 0$, we have $r \ge 0$. Combining both bounds gives

$$0 \le r \le \tfrac{1}{2}.$$

Recalling the definition of r, the stability condition is

$$\boxed{\frac{\alpha\,\Delta t}{(\Delta x)^2} \le \frac{1}{2}}$$

or equivalently

$$\boxed{\Delta t \le \frac{(\Delta x)^2}{2\alpha}}.$$

For the explicit forward-in-time, central-in-space finite-difference scheme applied to the 1D heat equation, von Neumann stability analysis yields the necessary and sufficient requirement

$$\frac{\alpha\,\Delta t}{(\Delta x)^2} \le \frac{1}{2},$$

i.e. $\Delta t \le (\Delta x)^2/(2\alpha)$. When this condition holds the scheme is stable; otherwise the numerical solution exhibits unbounded growth.

2 For the 1D Wave Equation[5, 2]

2.1 Governing Equation and discretization

The one-dimensional wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3}$$

where $u(x,t)$ is the displacement and c is the wave propagation speed.

We discretize the spatial and temporal domains as:

$$x_i = i\Delta x, \quad t^n = n\Delta t$$

and approximate the derivatives using central differences.
The second-order spatial derivative is approximated as:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \tag{4}$$

The second-order temporal derivative is approximated as:

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} \tag{5}$$

2.2 Discrete update equation

Substituting these approximations into the wave equation gives:

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} = c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \tag{6}$$

Rearranging, we obtain the explicit update formula:

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + r^2(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \tag{7}$$

where the non-dimensional parameter r is the *Courant number*:

$$r = \frac{c\Delta t}{\Delta x} \tag{8}$$

2.3 Von Neumann Stability Analysis

We assume a trial solution of the form:

$$u_i^n = G^n e^{ikx_i} \tag{9}$$

where G is the amplification factor and k is the wavenumber.

Substituting this into the finite difference scheme:

$$G^{n+1}e^{ikx_i} = 2G^n e^{ikx_i} - G^{n-1}e^{ikx_i} + r^2 G^n(e^{ikx_{i+1}} - 2e^{ikx_i} + e^{ikx_{i-1}}) \tag{10}$$

$$\Rightarrow G^{n+1} = 2G^n - G^{n-1} + 2r^2 G^n(\cos(k\Delta x) - 1) \tag{11}$$

2.4 Characteristic Equation

Assuming $G^n = \lambda^n$, we get:

$$\lambda^2 - 2\lambda[1 - r^2(1 - \cos(k\Delta x))] + 1 = 0 \tag{12}$$

For stability, the magnitude of all possible amplification factors must satisfy:

$$|\lambda| \leq 1 \tag{13}$$

2.5 Stability Condition

This requirement is satisfied only if:

$$r \leq 1 \tag{14}$$

or equivalently,

$$\boxed{\Delta t \leq \frac{\Delta x}{c}} \tag{15}$$

The above condition is known as the **Courant–Friedrichs–Lewy (CFL) condition** for the wave equation. It implies that the numerical domain of dependence must encompass the physical domain of dependence of the PDE. If $r > 1$, the numerical solution becomes unstable, leading to exponentially growing errors.

References

[1] Kurt Becker and Boris Kaus. Problem set: Finite difference example – 1d explicit heat equation. Course materials for GEOL 557 Numerical Modeling of Earth Systems, University of Texas at Austin, 2016. Available at: `https://www-udc.ig.utexas.edu/external/becker/teaching/557/problem_sets/problem_set_fd_explicit.pdf`.

[2] Universität Münster Institut für Theoretische Physik. Vorlesung numerische methoden i" — kapitel wave". Lecture notes, Universität Münster, Wintersemester 2016/2017, 2016. Available at: `https://www.uni-muenster.de/imperia/md/content/physik_tp/lectures/ws2016-2017/num_methods_i/wave.pdf`.

[3] Gerald Recktenwald. Ftcs solution to the heat equation. *Course ME*, 448:548, 2014.

[4] Padmanabhan Seshaiyer. Finite-difference method for the 1d heat equation. Course notes for Math 679, Fall 2012, George Mason University, Fall 2012. Available at: `https://math.gmu.edu/~pseshaiy/F12/m679/m679_F12_notes_hw1.pdf`.

[5] Unknown. Chapter 3.1-2. Course notes for CFD2003, University of Oklahoma, 2003. Available at: `https://twister.caps.ou.edu/CFD2003/Chapter3_1-2.pdf`.